

The discontinuous dynamics and non-autonomous chaos

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Abstract

A multidimensional chaos is generated by a special initial value problem for the non-autonomous impulsive differential equation. The existence of a chaotic attractor is shown, where density of periodic solutions, sensitivity of solutions and existence of a trajectory dense in the set of all orbits are observed. The chaotic properties of all solutions are discussed. An appropriate example is constructed, where the intermittency phenomenon is indicated. The results of the paper are illustrating that impulsive differential equations may play a special role in the investigation of the complex behavior of dynamical systems, different from that played by continuous dynamics.

1 Introduction and Preliminaries

The investigation of the complex behavior of dynamical systems, as well as the development of the methods of this investigation, has made a long way, starting with significant works [1]- [15].

It is natural *to discover a chaos* [1]-[3],[6], [7]-[12],[14, 15], and proceed by producing basic definitions and creating the theory. On the other hand, one can, following the prescriptions, *shape* an irregular process by inserting chaotic elements in a system which has regular dynamics otherwise (let's say is asymptotically stable, has a global attractor, etc). This approach to the problem also deserves consideration as it may allow for a more rigorous treatment of the phenomenon, and help develop new methods of investigation. Our results are of this type. Using the logistic map as a generator of moments of impulses in the multidimensional system, we observe the density of the periodic solutions, the sensitivity of solutions and the existence of a trajectory dense in the set of all solutions in a bounded region of the space.

The first mathematical definition of chaos was introduced by Li and Yorke [9]. They proved that if a map on an interval had a point of period three, then it had points of all periods. Moreover, there exists an uncountable *scrambled* subset of the interval. Devaney [4] gave an explicit definition of a chaotic invariant set in an attempt to clarify the notion of chaos. To the properties of *transitivity* and

sensitivity [12] he added the assumption that the periodic points are dense in the space. An intensive discussion of the definitions has continued during the last decades. It was shown in [16] that a map is Li-Yorke chaotic if and only if there is a two-point scrambled set. Paper [17] proves that chaos as defined by Devaney is stronger than that defined by Li-Yorke. In [18] it was shown that transitivity and density of periodic points imply sensitivity.

In this paper we concentrate on the topological components of the version proposed by Devaney. The special initial value problem is introduced, when the moments of the impulsive action are functionally dependent on the initial moment.

One of the most powerful tools of the chaos investigation is the conjugacy with the symbolic dynamics. Our results are based on the method, too. Consider the sequence space [4]

$$\Sigma_2 = \{s = (s_0 s_1 s_2 \dots) : s_j = 0 \text{ or } 1\}$$

with the metric

$$d[s, t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i},$$

where $t = (t_0 t_1 \dots) \in \Sigma_2$, and the shift map $\sigma : \Sigma_2 \rightarrow \Sigma_2$, such that $\sigma(s) = (s_1 s_2 \dots)$. The semidynamics (Σ_2, σ) is the symbolic dynamics [19].

The map is continuous, $\text{cardPer}_n(\sigma) = 2^n$, $\text{Per}(\sigma)$ is dense in Σ_2 , and there exists a dense orbit in Σ_2 .

We assume that there exist a homeomorphism S between Σ_2 and a set $\Lambda \subset I, I = [0, \bar{\omega}]$, where $\bar{\omega}$ is a fixed positive number, and a map $h : \Lambda \rightarrow \Lambda$, such that $S \circ h = \sigma \circ S$. That is h and σ are topologically conjugate. It is known that Σ_2 and Λ are Cantor sets: they are closed, perfect and totally disconnected [19]. Obviously, they are compact. Moreover, σ and h have sensitive dependence on the initial conditions, periodic points are dense in Σ_2 and Λ respectively, and

the maps are topologically transitive. We may also specify that for each $p \in \mathbb{N}$ there exists a solution with period p , and topological transitivity means having a dense trajectory.

One of the most popular examples of the map h is the logistic map $\mu x^2(1 - x)$, $\mu > 4$, considered on a subset of $[0, 1]$, [20].

The description of the main subject of our paper should begin with the discussion of the moments of impulses, as their generation is most important for the emergence of chaos.

For every $t_0 \in \Lambda$ one can construct a sequence $\kappa(t_0)$ of real numbers $\kappa_i, i \in \mathbb{Z}$, in the following way. If $i \geq 0$, then $\kappa_{i+1} = h(\kappa_i)$ and $\kappa_0 = t_0$. Let us show, how the sequence is defined for negative i . Denote $s^0 = S(t_0)$, $s^0 = (s_0^0 s_1^0 \dots)$. Consider elements $\underline{s} = (0 s_0^0 s_1^0 \dots)$, $\bar{s} = (1 s_0^0 s_1^0 \dots)$ of Σ_2 , such that $\sigma(\underline{s}) = \sigma(\bar{s}) = s^0$ and $\underline{t} = S^{-1}(\underline{s})$, $\bar{t} = S^{-1}(\bar{s})$. The homeomorphism implies that $h(\bar{t}) = h(\underline{t}) = t_0$. Set $h^{-1}(t_0)$ may consist of not more than two elements $\bar{t}, \underline{t} \in \Lambda$. Each of these two values can be chosen as $\kappa_{-1}(t_0)$. Obviously, one can continue the process to $-\infty$, choosing always one element from the set h^{-1} . We have finalized the construction of the sequence, and, moreover, it is proved that $\kappa(t_0) \subset \Lambda$. Thus, infinitely many sequences $\kappa(t_0)$ can be constructed for a given t_0 . However, each of this type of sequence is unique for an increasing i . Fix one of the sequences and define a sequence $\zeta(t_0) = \{\zeta_i\}$, $\zeta_i = i\bar{\omega} + \kappa_i, i \in \mathbb{Z}$. The sequence has a *periodicity property* if there exists $p \in \mathbb{N}$ such that $\zeta_{i+p} = \zeta_i + p\bar{\omega}, \forall i \in \mathbb{Z}$. If we denote by Π the set of all such sequences $\{\zeta_i\}, i \in \mathbb{Z}$, then a multivalued functional $w : I \rightarrow \Pi$ such that $\zeta(t_0) = w(t_0)$ is defined.

In what follows, we assume, without loss of generality, that $\bar{\omega} = 1$.

Let $J \subseteq \mathbb{R}$ be an open interval. We introduce the distance $\|\zeta(t_0) - \zeta(t_1)\|_J = \sup_{\zeta_i(t_0), \zeta_i(t_1) \in J} |\zeta_i(t_0) - \zeta_i(t_1)|$, and, if $\zeta(t) = \{\zeta_i(t)\}$ is a sequence from Π , and m is an integer, we denote $\zeta(t, m) = \{\zeta_{i+m}(t)\}, i \in \mathbb{N}$. More information about the

opportunities connected with this distance can be found in [21].

Next, let us consider some useful properties of the elements of Π . They are simple consequences of the topological conjugacy. The first one is a reformulation of the known property for symbolic dynamics [19].

LEMMA 1.1 *Assume that h is topologically conjugate to σ then the following assertions are valid:*

- (a) *for each $\zeta(t_0) \in \Pi$, an arbitrary small $\epsilon > 0$, and an arbitrary large positive number E there exists a sequence $\zeta(t_1) \in \Pi$ with the periodicity property such that $|\zeta(t_0) - \zeta(t_1)|_J < \epsilon$, where $J = (0, E)$*
- (b) *There exists a sequence $\zeta(t^*) \in \Pi$ such that for each $t_0 \in \Lambda$, and an arbitrary small $\epsilon > 0$, and an arbitrary large positive number E there exists an integer m such that $|\zeta(t_0) - \zeta(t^*, m)|_J < \epsilon$, where $J = (0, E)$.*

Proof: (a) Fix numbers $t_0 \in \Lambda, \epsilon > 0, E > 0$, and denote $S(t_0) = s^0 = (s_0^0 s_1^0 \dots)$. Since S is a homeomorphism and Σ_2 is compact, there exists a number $\delta > 0$ such that $|S^{-1}(s^1) - S^{-1}(s^2)| < \epsilon$ if $d[s^1, s^2] < \delta$, where $s^1, s^2 \in \Sigma_2$.

Next we define a periodic sequence $s = (s_0 s_1 s_3 \dots)$ from Σ_2 to satisfy ϵ and E . Take a number $l \in \mathbb{N}$ such that $l > E$ and $\frac{1}{2^{l-1}} < \delta$. Assume that $s_i = s_i^0, i = 0, 1, \dots, 2l-1$, and $s_{i+2l} = s_i, i \in \mathbb{Z}$. Consider the sequence $\kappa_i(t_0) = h^i(t_0), i \geq 0$. Since $h^i(t_0) = S^{-1} \circ \sigma^i \circ S$, and $d[\sigma^i s^0, \sigma^i s] < \delta, i = 0, 1, \dots, l$, we have that $|h^i(t_0) - h^i(S^{-1}(s))| < \epsilon$. So, if we denote $t = S^{-1}(s)$ then $|\kappa_i(t_0) - \kappa_i(t)| < \epsilon, i = 0, 1, \dots, l$. In other words, $\|\zeta(t_0) - \zeta(t)\|_J < \epsilon$, where $J = (0, E)$. The assertion is proved.

(b). Consider the sequence $s^* \in \Sigma_2$ such that

$$s^* = \underbrace{01}_{1 \text{ element blocks}} \mid \underbrace{00011011}_{2 \text{ element blocks}} \mid \dots,$$

that is s^* is constructed by successively listing all blocks of 0's and 1's of length 1, then length 2, etc. Sequence s^* is dense in Σ_2 [4]. Denote $t^* = S^{-1}(s^*)$. Let

us fix $t_0 \in \Lambda, \epsilon > 0$ and $E > 0$. Similarly to the previous proof, fix a number $\delta > 0$ such that $|S^{-1}(s^1) - S^{-1}(s^2)| < \epsilon$ if $d[s^1, s^2] < \delta$, where $s^1, s^2 \in \Sigma_2$. Take $l \in \mathbb{N}$ such that $l > E + 1$ and $\frac{1}{2^{l-1}} < \delta$. We can find $2l$ - block $s_m^* s_{m+1}^* \dots s_{m+2l}^*$ such that i -th element of $S(t_0), i = 0, 1, \dots, 2l$, is equal to s_{i+m}^* . One can see that $\|\zeta(t_0) - \zeta(t^*, m)\|_J < \epsilon$, where $J = (0, E)$. The lemma is proved.

The following special initial value problem for the impulsive differential equation,

$$\begin{aligned} z'(t) &= Az(t) + f(z), t \neq \zeta_i(t_0), \\ \Delta|_{t=\zeta_i(t_0)} &= Bz(\zeta_i(t_0)) + W(z(\zeta_i(t_0))), \\ z(t_0) &= z_0, (t_0, z_0) \in \Lambda \times \mathbb{R}^n, \end{aligned} \tag{1}$$

where $z \in \mathbb{R}^n, t \in \mathbb{R}$, and $\zeta(t_0) = w(t_0)$, is the object, which will be mainly discussed in our paper.

We shall need the following basic assumptions for the system:

(C1) A, B are $n \times n$ constant real valued matrices, $\det(\mathcal{I} + B) \neq 0$, where \mathcal{I} is $n \times n$ identical matrix;

(C2) for all $x_1, x_2 \in \mathbb{R}^n$ functions $f(x, y) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n, W : \mathbb{R}^n \rightarrow \mathbb{R}^n$, satisfy

$$\|f(x_1) - f(x_2)\| + \|W(x_1) - W(x_2)\| \leq L\|x_1 - x_2\|, \tag{2}$$

where $L > 0$ is a constant;

(C3)

$$Bx + W(x) \neq 0, \forall x \in \mathbb{R}^n; \tag{3}$$

(C4) functions f and W are uniformly bounded so that

$$\sup_{x \in \mathbb{R}^n} \|f(x)\| + \sup_{x \in \mathbb{R}^n} \|W(x)\| = M_0 < \infty. \tag{4}$$

We considered the systems of form (1) in [22] in the investigation of blood pressure distribution.

In the present paper the sensitivity, transitivity and existence of dense periodic solutions of nonlinear problem (1) are considered. We also prove the existence of the chaotic attractor, which consists of all solutions bounded on the real axis and discuss the period-doubling cascade for the problem.

All the results are rigorously verified using the results of the theory of impulsive differential equations [23]-[31].

While many theorems of the theory of impulsive differential equations are obtained by analogy with the assertions which already exist for the systems with continuous solutions, the results of the present work do not have direct predecessors from the continuous dynamics.

The investigation is inspired by the discontinuous dynamics of the neural information processing in the brain, information communication, and population dynamics [32]-[38]. While there are many interesting papers concerned with the complex behavior generated by impulses, the rigorous theory of chaotic impulsive systems remains far from being complete. Our goal is to develop further the theoretical foundations of this area of research.

The paper is organized in the following manner. The existence of the attractor is under discussion in the next section. In section 3 we consider the main subject of the paper: ingredients of chaos, the chaotic attractor, the period-doubling cascade and an example.

2 An attractor consisting of the solutions bounded on \mathbb{R}

In this section we discuss the existence of bounded and periodic solutions of the initial value problem, and the existence of an attractor consisting of the bounded

on \mathbb{R} solutions.

Let us start with a system of a more general form than (1).

Fix a sequence $\theta = \{\theta_i\}, i \in \mathbb{Z}$, such that $i \leq \theta_i < i + 1, i \in \mathbb{Z}$, and consider the following initial value problem

$$\begin{aligned} z'(t) &= Az(t) + f(z), t \neq \theta_i, \\ \Delta|_{t=\theta_i} &= Bz(\theta_i) + W(z(\theta_i)), \\ z(\theta_0) &= z_0, z_0 \in \mathbb{R}^n, \end{aligned} \tag{5}$$

where matrices A and B , functions f and W are the same as in (1).

In order to define the solutions of the impulsive systems we need the following spaces of discontinuous functions.

A left continuous function $z(t)$ is from a set of functions $\mathcal{PC}^1(\theta, \mathbb{R})$, if:

- (i) it has discontinuities only at points $\theta_i, i \in \mathbb{Z}$, and they are of the first kind;
- (ii) the derivative $z'(t)$ exists at each point $t \in \mathbb{R} \setminus \{\theta_i\}$, and at points $\theta_i, i \in \mathbb{Z}$, the left-sided derivative exists.

A function $z(t) \in \mathcal{PC}^1(\theta, \mathbb{R})$ is a solution of (5) if:

- (1) the differential equation is satisfied for $z(t)$ on $\mathbb{R} \setminus \{\theta_i\}$, and it holds for the left derivative of $z(t)$ at every point $\theta_i, i \in \mathbb{Z}$.
- (2) the jumps equation is satisfied by $z(t)$ for every $i \in \mathbb{Z}$.

It is known [24, 25] that under the above mentioned conditions the solution of (5) exists, is unique on \mathbb{R} , and it is an element of $\mathcal{PC}^1(\theta, \mathbb{R})$.

We shall also need the following set of functions.

A left continuous function $z(t) : [\theta_0, \infty) \rightarrow \mathbb{R}^n$ is from a set of functions $\mathcal{PC}^1(\theta, [\theta_0, \infty))$, if:

- (i) it has discontinuities only at points $\theta_i, i \geq 0$, and these discontinuities are of the first kind;
- (ii) the derivative $z'(t)$ exists at each point $t \in [\theta_0, \infty) \setminus \{\theta_i\}$, and at points $\theta_i, i \geq 0$, the left-sided derivative exists;

A solution $z(t)$ of (5) on $[\theta_0, \infty)$ is a function $z(t) \in \mathcal{PC}^1(\theta, [\theta_0, \infty))$ such that:

- (1) the differential equation is satisfied for $z(t)$ on $[\theta_0, \infty) \setminus \{\theta_i\}$, and it holds for the left derivative of $z(t)$ at every point $\theta_i \geq 0$.
- (2) the jumps equation is satisfied by $z(t)$ for every $i \geq 0$.

Let us denote $Z(t, s)$ the transition matrix of the associated with (5) linear homogeneous system

$$\begin{aligned} z'(t) &= Az(t), t \neq \theta_i \\ \Delta z|_{t=\theta_i} &= Bz(\theta_i). \end{aligned} \tag{6}$$

We may assume that:

- (C5) the matrices A and B commute, the real parts of all eigenvalues of the matrix $A + \ln(\mathcal{I} + B)$ are negative.

Condition (C5) implies, [25], Theorem 34, that there exist two positive numbers N and ω , which do not depend on θ ,

$$\|Z(t, s)\| \leq Ne^{-\omega(t-s)}, t \geq s. \tag{7}$$

Obviously, that condition for the matrices to commute is strong. One can obtain the exponential decay of the transition matrix using other conditions on matrices A and B , but this would lengthen the paper significantly, and, moreover, decrease the clarity of the proof.

Assume additionally that

$$(C6) \quad NL[\frac{2}{\omega} + \frac{e^\omega}{1-e^{-\omega}}] < 1.$$

The proof of following two Theorems 2.1 and 2.2 replicate those of Theorems 37 and 89 from [25], respectively.

THEOREM 2.1 *If conditions (C1), (C2), (C4) – (C6) are valid, then:*

1. *there exists a unique bounded solution $z(t)$ of (5) from $\mathcal{PC}^1(\theta, \mathbb{R})$,*

$$\text{and } \|z(t)\| < NM_0[\frac{1}{\omega} + \frac{e^\omega}{1-e^{-\omega}}] \text{ for all } t \in \mathbb{R};$$

2. *if there exists a number $p \in \mathbb{N}$, such that $\theta_{i+p} = \theta_i + p, i \in \mathbb{Z}$, then the bounded solution $z(t)$ has the period p ;*

We may assume that

$$(C7) \quad -\omega + NL + \ln(1 + NL) < 0.$$

Let $z(t)$ be the bounded solution, which exists by Theorem 2.1, and $z_1(t)$ be another solution of (5). The following theorem is valid.

THEOREM 2.2 *Assume that conditions (C1), (C2), (C4) – (C7) are fulfilled.*

Then, for each $t, s \in \mathbb{R}, t \geq s$,

$$\|z(t) - z_1(t)\| \leq \|z(s) - z_1(s)\|(1 + NL)e^{(-\omega + NL + \ln(1 + NL))(t-s)}. \quad (8)$$

That is, the bounded solution $z(t)$ attracts all solutions of (5).

In what follows we denote $z(t, \xi, v), \xi \in \mathbb{R}, v \in \mathbb{R}^n$, a solution of (1) with $t_0 = \xi, z_0 = v$, and assume that all conditions (C1) – (C7) are fulfilled.

Assume that in (5) $\theta = \zeta(t_0), t_0 \in \Lambda$. Then Theorem 2.1 implies that for each $t_0 \in \Lambda$ there exists a unique bounded on \mathbb{R} solution of (1). We denote the

solution by $z(t, t_0)$. Moreover, if $\zeta(t_0)$ has the periodicity property with period p , then the solution $z(t, t_0)$ is p -periodic. Denote the periodic solutions as $\phi(t, t_0)$.

By Theorem 2.2 the bounded solution $z(t, t_0)$ attracts all solutions of (1) with the same points of discontinuity. That is, if $z(t, t_0, z_0)$ is another solution of (1), then

$$\|z(t, t_0) - z(t, t_0, z_0)\| \leq \|z(t_0, t_0) - z_0\|(1 + NL)e^{(-\omega + NL + \ln(1 + NL))(t - t_0)}, \quad (9)$$

for all $t \geq t_0$. Finally, we remark that there exist infinitely many periodic solutions $\phi(t, t_0), t_0 \in \Lambda$. More precisely, for each $p \in \mathbb{N}$ there exists $t_0 \in \Lambda$ such that $z(t, t_0)$ is a p -periodic solution. These periodic solutions are different for different p since by condition (C3) they have sequences of discontinuity points which do not intersect.

Denote $\mathcal{PC} = \{z(t, t_0, z_0) : t_0 \in \Lambda, z_0 \in \mathbb{R}^n\}$, and let $\mathcal{PCB} \subset \mathcal{PC}$ be the subset of all solutions bounded on \mathbb{R} . That is, $\mathcal{PCB} = \{z(t, t_0) : t_0 \in \Lambda\}$. On the basis of above made discussion we may say that the bounded set \mathcal{PCB} is an attractor with the basin \mathcal{PC} .

3 The chaos

In this section we introduce the topological ingredients of the chaos for the dynamics of (1) and find the conditions for the existence of these ingredients. Moreover, we will prove that \mathcal{PCB} is the chaotic attractor, and discuss the period-doubling cascade for the problem.

3.1 The chaotic attractor existence

DEFINITION 3.1 *We say that (1) is sensitive on \mathcal{PCB} if there exist positive real numbers ϵ_0, ϵ_1 such that for each $t_0 \in \Lambda$, and for every $\delta > 0$ one could find a number $t_1 \in \Lambda, |t_0 - t_1| < \delta$, and an interval Q from $[t_0, \infty)$ with length no*

less than ϵ_1 such that $\|z(t, t_0) - z(t, t_1)\| \geq \epsilon_0, t \in Q$, and there are no points of discontinuity of $z(t, t_0), z(t, t_1)$ in Q .

To introduce following definitions of density we need the concept of "closeness" for piecewise continuous functions. Different types of metrics and topologies for discontinuous functions are described in [23], [25]-[31], [21],[39],[40]. In what follows we are going to use the concept from [39].

Let us fix $t_0, t_1 \in I$ and an interval $J \subset [t_0, \infty) \cap [t_1, \infty)$. We say that a function $\xi(t) \in \mathcal{PC}^1(\zeta(t_0), \mathbb{R}_{\zeta(t_0)})$ is ϵ -equivalent to a function $\psi(t) \in \mathcal{PC}^1(\zeta(t_1), \mathbb{R}_{\zeta(t_1)})$ on J and write $\xi(t)(\epsilon, J)\psi(t)$ if $\|\zeta(t_0) - \zeta(t_1)\|_J < \epsilon$ and $\|\xi(t) - \psi(t)\| < \epsilon$ for all t from J such that $t \notin \cup_{\zeta_i(t_0), \zeta_i(t_1) \in J} [\widehat{\zeta_i(t_0)}, \widehat{\zeta_i(t_1)}]$, where $[\widehat{a}, \widehat{b}]$, $a, b \in \mathbb{R}$, means an oriented interval, that is $[\widehat{a}, \widehat{b}] = [a, b]$ if $a \leq b$, and $[\widehat{a}, \widehat{b}] = [b, a]$, if otherwise. The equivalence of two piecewise continuous functions when ϵ is small means, roughly speaking, that they have close discontinuity points, and the values of the functions are close at points that do not lie on intervals between the neighbor discontinuity points of these functions.

DEFINITION 3.2 *The set of all periodic solutions $\phi(t) = \phi(t, t_0), t_0 \in \Lambda$, of (1) is called dense in \mathcal{PCB} if for every solution $z(t, t_1), t_1 \in \Lambda$, and each $\epsilon > 0, E > 0$, there exists a periodic solution $\phi(t, t^*), t^* \in \Lambda$, and an interval $J \subset \mathbb{R}_{\zeta(t_1)}$ with length E such that $\phi(t)(\epsilon, J)z(t)$.*

DEFINITION 3.3 *A bounded solution $z(t) = z(t, t^*), t^* \in \Lambda$, of (1) is called dense in the set of all orbits of \mathcal{PCB} if for every solution $z_1(t) = z(t, t_1), t_1 \in \Lambda$, of (1), and each $\epsilon > 0, E > 0$, there exists an interval J with length E and a real number ξ such that $z(t + \xi)(\epsilon, J)z_1(t)$.*

DEFINITION 3.4 *The attractor \mathcal{PCB} is chaotic if*

1. (1) is sensitive in \mathcal{PCB} ;

2. the set of all periodic solutions $\phi(t, t_0), t_0 \in \Lambda$, is dense in \mathcal{PCB} ;

3. there exists a solution $z(t, t_0), t_0 \in \Lambda$, which is dense in \mathcal{PCB} .

THEOREM 3.1 *Assume that conditions (C1) – (C7) are fulfilled. Then the set of all periodic solutions $\phi(t, t_0), t_0 \in \Lambda$, of (1) is dense in \mathcal{PCB} .*

Proof: Let us fix $t_1 \in \Lambda$ and $E, \epsilon > 0$. By Lemma 1.1 (a) for an arbitrary large number \tilde{T} there exists a sequence $\zeta(t_0) \in \Pi$ with periodic property such that $\|\zeta(t_1) - \zeta(t_0)\|_Q < \epsilon$, where $Q = (t_1, t_1 + \tilde{T} + E)$. We shall find the number \tilde{T} sufficiently large such that the bounded solution $z(t) = z(t, t_1)$ is ϵ -equivalent to $\phi(t, t_0)$ on $J = (t_1 + \tilde{T}, t_1 + \tilde{T} + E)$. Denote $M_1 = NM_0(\frac{1}{\omega} + \frac{e^\omega}{1-e^{-\omega}})$.

Let also $Z_1(t, s) = Z(t, s, \zeta(t_1))$ and $Z_2(t, s) = Z(t, s, \zeta(t_0)), t \geq s$, be the transition matrices. We have that [25]

$$\begin{aligned} z(t) &= Z_1(t, 1)z(1) + \int_1^t Z_1(t, s)f(z(s))ds + \sum_{1 \leq \zeta_i < t} Z_1(t, \zeta_i(t_1))W(z(\zeta_i(t_1))), \\ \phi(t) &= Z_2(t, 1)\phi(1) + \int_1^t Z_2(t, s)f(\phi(s))ds + \sum_{1 \leq \zeta_i < t} Z_2(t, \zeta_i(t_0))W(\phi(\zeta_i(t_0))). \end{aligned}$$

It is difficult to evaluate the difference between $z(t)$ and $\phi(t)$ using the last two expressions since the moments of discontinuities do not coincide. For this reason let us apply the method of B -equivalence developed in papers [26]-[31].

Let us consider the following system

$$\begin{aligned} v'(t) &= Av(t) + f(v), t \neq \zeta_i(t_0), \\ \Delta v|_{t=\zeta_i(t_0)} &= Bv(\zeta_i(t_0)) + W(v(\zeta_i(t_0))) + W_i^1(v(\zeta_i(t_0))), \end{aligned} \quad (10)$$

and the system

$$\begin{aligned} z'(t) &= Az(t) + f(z), t \neq \zeta_i(t_1), \\ \Delta z|_{t=\zeta_i(t_1)} &= Bz(\zeta_i(t_1)) + W(\zeta_i(t_1)), \end{aligned} \quad (11)$$

where t_0, t_1 are the numbers under discussion. Assuming, without loss of generality, that $\zeta_j(t_0) \leq \zeta_j(t_1)$ for all j , we introduce

$$\begin{aligned}
W_i^1(z) &= (\mathcal{I} + B) \left[(e^{A(\zeta_i(t_1) - \zeta_j(t_0))} - \mathcal{I})z + \int_{\zeta_j(t_0)}^{\zeta_i(t_1)} e^{A(\zeta_i(t_1) - s)} f(z(s)) ds \right] + \\
&W((\mathcal{I} + B)[e^{A(\zeta_i(t_1) - \zeta_j(t_0))} z + \int_{\zeta_j(t_0)}^{\zeta_i(t_1)} e^{A(\zeta_i(t_1) - s)} f(z(s)) ds]) - \\
&\int_{\zeta_j(t_0)}^{\zeta_i(t_1)} e^{A(\zeta_i(t_1) - s)} f(z_1(s)) ds - W(z),
\end{aligned}$$

where $z(t), z_1(t)$ are the solutions of

$$z'(t) = Az(t) \quad (12)$$

such that $z(\zeta_i(t_0)) = z$ and $z_1(\zeta_i(t_1)) = z(\zeta_i(t_1)+)$. One can easily verify that $M_2 = \sup_{\|z\| \leq M_1, i \in \mathbb{Z}} \|W_i^1(z)\| < \infty$. Systems (10) and (11) are B -equivalent [26]-[31]. That is their two solutions with the same initial data coincide in their common domain if only $t \notin (\widehat{\zeta_i(t_0)}, \widehat{\zeta_i(t_1)}], i \in \mathbb{Z}$.

So, if $v(t), v(1) = z(1)$, is the solution of (10), then $v(t) = z(t)$ for all $t \notin (\widehat{\zeta_i(t_0)}, \widehat{\zeta_i(t_1)}], i \in \mathbb{Z}$.

For $v(t)$ we have that

$$\begin{aligned}
v(t) &= Z_2(t, 1)v(1) + \int_1^t Z_2(t, s)f(v(s))ds + \\
&\sum_{1 \leq \zeta_i < t} Z_2(t, \zeta_i(t_0))[W(v(\zeta_i(t_0))) + W_1(v(\zeta_i(t_0)))].
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|\phi(t) - v(t)\| &\leq \|\phi(1) - v(1)\| \|Z_2(t, 1)\| + \int_1^t \|Z_2(t, s)\| L \|\phi(s) - v(s)\| ds + \\
&\sum_{1 \leq \zeta_j(t_0) < t} \|Z_2(t, \zeta_j(t_0))\| L \|\phi(\zeta_j(t_0)) - v(\zeta_j(t_0))\| + \sum_{1 \leq \zeta_j(t_0) < t} \|Z_2(t, \zeta_j(t_0))\| \|W_1(v(\zeta_i(t_0)))\| \leq \\
&2M_1N + M_2 \frac{e^\omega}{1 - e^{-\omega}} + \int_1^t N e^{-\omega(t-s)} L \|z(s) - v(s)\| ds + \\
&\sum_{1 \leq \zeta_j < t} N e^{-\omega(t-\zeta_j(t_0))} L \|v(\zeta_j(t_0)) - v(\zeta_j(t_0))\|.
\end{aligned}$$

Now, applying the analogue of Gronwall-Bellman Lemma [25] for discontinuous functions, one can find that

$$\begin{aligned} \|z(t) - v(t)\| &\leq (2M_1N + M_2 \frac{e^\omega}{1 - e^{-\omega}}) e^{(-\omega + NL)(t-1)} \prod_{1 \leq \zeta_j < t} (1 + NL) \leq \\ &(2M_1N + M_2 \frac{e^\omega}{1 - e^{-\omega}}) e^{(-\omega + NL + \ln(1 + NL))(t-1)}. \end{aligned} \quad (13)$$

Inequality (13) implies that $\|z(t) - v(t)\| < \epsilon$ if $t > \tilde{T}$, $t \notin [\widehat{\zeta_i(t_0)}, \widehat{\zeta_i(t_1)}]$, $i \in \mathbb{Z}$, where

$$\tilde{T} = 1 + \frac{\ln(\frac{\epsilon}{2M_1N + M_2 e^\omega (1 - e^{-\omega})^{-1}})}{-\omega + NL + \ln(1 + NL)},$$

(we may assume that $\epsilon < 2M_1$). That is why, $z(t)(\epsilon, J)\phi(t)$ if $J = (t_1 + \tilde{T}, t_1 + \tilde{T} + E)$. The theorem is proved.

THEOREM 3.2 *Assume that conditions (C1) – (C7) are fulfilled. Then there exists a solution of (1) $z(t, t^*)$ dense in \mathcal{PCB} .*

Proof: Fix positive E, ϵ . By Lemma 1.1 (b) there exists $t^* \in \Lambda$ such that $\zeta(t^*)$ is dense in Π . Denote $z_*(t) = z(t, t^*)$. Let us prove that $z_*(t)$ is the dense solution.

Consider an arbitrary solution $z(t) = z(t, t_0) \in \mathcal{PCB}$. Consider an interval $J_1 = (0, E_1)$, where E_1 is an arbitrarily large positive number. By Lemma 1.1 (b) there exists a natural m such that

$$\|\zeta(t_1) - \zeta(t^*, m)\|_{J_1} < \epsilon. \quad (14)$$

We have that

$$\begin{aligned} z_*(t + m) &= e^{A(t+m-1-m)} z_*(1 + m) + \int_{1+m}^{t+m} e^{A(t+m-u)} f(z_*(u)) du + \\ &\sum_{1+m \leq \zeta_i(t_0) < t+m} e^{A(t+m-\zeta_i(t_0))} W(z_*(\zeta_i(t_0))) = e^{A(t-1)} z_*(1 + m) + \\ &\int_1^t e^{A(t-u)} f(z_*(u + m)) du + \sum_{1+m \leq \zeta_i(t_0) < t+m} e^{A(t+m-\zeta_i(t_0))} W(z_*(\zeta_i(t_0))). \end{aligned}$$

and

$$z_1(t) = e^{A(t-1)}z_1(1) + \int_1^t e^{A(t-u)}f(z_1(u))du + \sum_{s \leq \zeta_i(t_1) < t} e^{A(t-\zeta_i(t_1))}W(z_1(\zeta_i(t_1))).$$

Now, using the last two formulas, similarly to proof of Theorem 3.1, using (14) and the B -equivalence technique, we can find a sufficiently large number $E_1 > 2E$, and a natural number m such that $z_*(t+m)$ and $z_1(t)$ are ϵ -equivalent on $J = (E_1/2, E_1)$. The theorem is proved.

Denote $\overline{m} = \max_{|u| \leq 1} \|e^{Au}\|$, $\underline{m} = \min_{|u| \leq 1} \|e^{Au}\|$.

Condition (C3) implies that

$$\eta = \min_{\|x\| \leq M_1} (Bx + W(x)) > 0.$$

Fix a number $q \geq 3$, such that $\frac{1}{q} < \frac{2m}{3m}$.

We shall need the following assumption.

$$(C8) \quad L < \frac{[\frac{2m}{3m} - \frac{1}{q}]m\eta}{2M_1(\overline{m} + \underline{m})}.$$

THEOREM 3.3 *Assume that conditions (C1) – (C8) are fulfilled. Then (1) is sensitive on \mathcal{PCB} .*

Proof: Fix a solution $z(t) = z(t, t_0)$, $t_0 \in \Lambda$, and a positive δ .

Let $S(t_0) = s^0 = (s_0^0, s_1^0 \dots)$. Fix a number $t_1 \in \Lambda$ such that $S(t_1) = s^1 = (s_0^1, s_1^1 \dots, s_{\tilde{n}-1}^1, s_{\tilde{n}}^1, s_{\tilde{n}+1}^0, s_{\tilde{n}+2}^0, \dots)$, $s_{\tilde{n}}^1 \neq s_{\tilde{n}}^0$, for some $\tilde{n} > 0$. We have that

$$d[\sigma^i s^0, \sigma^i s^1] = \begin{cases} \frac{1}{2^{\tilde{n}-i}} & \text{if } 0 \leq i \leq \tilde{n}, \\ 0 & \text{if } i > \tilde{n}. \end{cases}$$

Assume that $\tilde{n} \geq 3$, \tilde{n} is sufficiently large for $|t_0 - t_1| = |S^{-1}(s^0) - S^{-1}(s^1)| < \delta$.

Now, denote $z_1(t) = z(t, t_1)$ the solution of (1).

Since S is a homeomorphism and set Σ_2 is compact, for a given i , $0 \leq i \leq \tilde{n}$, the set

$$P_i = \{(\bar{s}, \tilde{s}) \in \Sigma_2 \times \Sigma_2 : d[\bar{s}, \tilde{s}] \geq \frac{1}{2^{\tilde{n}-i}}\}$$

is compact, and

$$\min_{(\bar{s}, \tilde{s}) \in P_i} |S^{-1}(\bar{s}) - S^{-1}(\tilde{s})| = \mu_i > 0,$$

$P_{i+1} \subseteq P_i, \mu_{i+1} \geq \mu_i, 0 \leq i < \tilde{n}$. Fix $i_0 = \tilde{n} - 2$. Then $|\kappa_i(t_0) - \kappa_i(t_1)| \geq \mu_{i_0}$ if $i = i_0, i_0 + 1$.

Similarly, we also have that there exists a positive number $\mu_0 < 1$ such that $|\kappa_i(t_0) - \kappa_i(t_1)| \leq \mu_0$ if $0 \leq i < \tilde{n}$.

Without loss of generality assume that $\kappa_i(t_0) < \kappa_i(t_1)$ for all i . Thus, there is a number k among $i_0, i_0 + 1$ such that $\kappa_k(t_1) - \kappa_k(t_0) > \mu_{i_0}$ and $\kappa_k(t_0) - \kappa_{k-1}(t_1) \geq \frac{1}{2}(1 - \mu_0)$.

One can easily check that (C8) implies that $\nu_1 = \frac{2}{3} \frac{\underline{m}\eta}{\underline{m}} - 2LM_1 > 0$ and $\nu_2 < \nu_1$, where $\nu_2 = \frac{2\overline{m}LM_1}{\underline{m}} + \frac{1}{q}\eta$.

We shall show that constants ϵ_0, ϵ_1 for Definition 3.1 can be taken equal to $\epsilon_0 = \frac{1}{q}\underline{m}\eta, \epsilon_1 = \min\{\mu_{i_0}, \frac{1}{2}(1 - \mu_0)\}$.

Assume that $\|z(\zeta_k(t_0)) - z_1(\zeta_k(t_0))\| < \nu_1$.

Then, for $t \in [\zeta_k(t_0), \zeta_k(t_1)]$,

$$z(t) = e^{A(t-\zeta_k(t_0))}(\mathcal{I}+B)z((\zeta_k(t_0))) + \int_{\zeta_k(t_0)}^t e^{A(t-s)}f(z(s))ds + e^{A(t-\zeta_k(t_0))}W(z((\zeta_k(t_0)))),$$

$$z_1(t) = e^{A(t-\zeta_k(t_0))}z_1((\zeta_k(t_0))) + \int_{\zeta_k(t_0)}^t e^{A(t-s)}f(z_1(s))ds.$$

We have that

$$\begin{aligned} \|z(t) - z_1(t)\| &= \|e^{A(t-\zeta_k(t_0))}[Bz((\zeta_k(t_0))) + W(z((\zeta_k(t_0))))] + \\ &e^{A(t-\zeta_k(t_0))}[z((\zeta_k(t_0))) - z_1((\zeta_k(t_0)))] + \int_{\zeta_k(t_0)}^t e^{A(t-s)}(f(z(s)) - f(z_1(s)))ds\| \geq \\ &\underline{m}\eta - \overline{m}(\nu_1 + 2LM_1) \geq \epsilon_0. \end{aligned}$$

If $\|z(\zeta_k(t_0)) - z_1(\zeta_k(t_0))\| > \nu_2$, then, for $t \in [\zeta_{k-1}(t_1), \zeta_k(t_0)]$,

$$z(t) = e^{A(t-\zeta_k(t_0))}z((\zeta_k(t_0))) + \int_{\zeta_k(t_0)}^t e^{A(t-s)}f(z(s))ds,$$

$$z_1(t) = e^{A(t-\zeta_k(t_0))} z_1((\zeta_k(t_0))) + \int_{\zeta_k(t_0)}^t e^{A(t-s)} f(z_1(s)) ds.$$

and

$$\|z(t) - z_1(t)\| \geq \underline{m}\nu_2 - \overline{m}2LM_1 = \epsilon_0.$$

The theorem is proved.

On the basis of Theorems 3.1-3.3 we can conclude that the following theorem is valid

THEOREM 3.4 *If conditions (C1) – (C8) are fulfilled, then \mathcal{PCB} is a chaotic attractor.*

Next, we are going to show that the attractiveness arranged by Theorem 2.2 causes an amusing phenomena that the chaotic properties being attributed to all solutions of (1), not only those from \mathcal{PCB} .

3.2 The chaos of the initial value problem

Let us first introduce the following definitions.

DEFINITION 3.5 *We say that (1) is sensitive on Λ if there exist positive real numbers κ_0, κ_1 such that for every solution $z(t) = z(t, t_0, z_0), t_0 \in \Lambda$, of (1) and for each $\delta > 0, H > 0$, one could find a number $t_1 \in \Lambda$, such that $|t_0 - t_1| < \delta$, and an interval Q from $[t_0, \infty)$ with length no less than κ_1 such that $\|z_1(t) - z(t)\| \geq \kappa_0, t \in Q$, for all solutions $z_1(t) = z(t, t_1, z_1), \|z_0 - z_1\| < H$, of (1), and there are no points of discontinuity of $z_1(t)$ and $z(t)$ on Q .*

DEFINITION 3.6 *The set of all periodic solutions $\phi(t, t_0), t_0 \in \Lambda$, of (1) is called dense in \mathcal{PC} if for every solution $z(t) = z(t, t_1, z_0), t_1 \in \Lambda$, of the initial value problem, and each $\epsilon > 0, E > 0$, there exists a periodic solution $\phi(t)$ and an interval $J \subset \mathbb{R}_{t_1}$ with length E such that $\phi(t)(\epsilon, J)z(t)$.*

DEFINITION 3.7 A solution $z(t) = z(t, t^*, z_0), t^* \in \Lambda, t \geq t^*$, of (1) is called dense in the set of all orbits of \mathcal{PC} if for every solution $z_1(t) \in \mathcal{PC}$, and each $\epsilon > 0, E > 0$, there exists an interval J with length E and a real number ξ such that $z(t + \xi)(\epsilon, J)z_1(t)$.

DEFINITION 3.8 Problem (1) is chaotic on Λ if

(i) it has the chaotic attractor \mathcal{PCB} ;

(ii) it is sensitive on \mathcal{PC} ;

(iii) the set of all periodic solutions $\phi(t, t_0), t_0 \in \Lambda$, is dense in \mathcal{PC} ;

(iv) there exists a solution $z(t, t_0, z_0), t_0 \in \Lambda$, of (1), which is dense in the set of all orbits of \mathcal{PC} ;

THEOREM 3.5 Assume that conditions (C1) – (C8) are fulfilled. Then (1) is chaotic on Λ .

Proof: Condition (i) of Definition 3.8 is verified in Theorem 3.4. Using the attractiveness of \mathcal{PCB} one can verify that parts (ii), (iii) and (iv) are also valid. Indeed, let us show first that (iv) is true. We will show that the solution $z_*(t) = z(t, t^*)$ dense in \mathcal{PCB} by Theorem 3.2 is also dense in \mathcal{PC} . Fix a solution $z(t) = z(t, t_0, z_0) \in \mathcal{PC}$ and a positive ϵ . The solution $z(t)$ is attracted by the bounded solution $z(t, t_0) \in \mathcal{PCB}$, $z(t) \neq z(t, t_0)$, so that (9) is true. Hence, one can find a positive number T such that $\|z(t, t_0, z_0) - z(t, t_0)\| < \epsilon/2$ if $t \geq T$. In the proof of Theorem 3.2 it was shown that there exists an arbitrarily large number E_1 , where we may assume that $E_1 > 2T$, and a natural number m such that $z_*(t+m)$ is $\epsilon/2$ -equivalent to $z(t, t_0)$ on the interval $J = (E_1/2, E_1)$. Now, it is easy to see that $z_*(t+m)(\epsilon, J)z(t)$. So, (iv) is proven. Part (iii) can be verified in a similar way. Now consider condition (ii). To prove it, we will follow the proof and the

notation of Theorem 3.2 and we will show that the numbers in Definition 3.5 can be as $\kappa_0 = \epsilon_0/2$ and $\kappa_1 = \epsilon_1$, where ϵ_0 and ϵ_1 are the numbers used in the proof of Theorem 3.2. Consider a solution $z(t, t_0, z_0) \in \mathcal{PC}$. From (9) it follows that for a fixed $H > 0$ there exists a number $T = T(H, z_0)$ such that if $t \geq T$, then $\|z(t, \xi, z_1) - z(t, \xi)\| < \epsilon_0/4$, where $\xi \in \Lambda$ is arbitrary, and $\|z_1 - z_0\| < H$. Analyzing the proof of Theorem 3.2 one can readily observe that the number \tilde{n} , which satisfies all other demands of the proof, can be chosen greater than $T + 2$. Consequently, the number $t_1 \in \Lambda$ corresponding to \tilde{n} satisfies $|t_1 - t_0| < \delta$, and $\|z(t, t_0) - z(t, t_1)\| \geq \epsilon_0, t \in Q$. Hence, $\|z(t, t_0, z_0) - z(t, t_1, z_1)\| \leq \|z(t, t_0) - z(t, t_1)\| - \|z(t, t_0, z_0) - z(t, t_0)\| - \|z(t, t_1, z_1) - z(t, t_1)\| \geq \epsilon_0/2 = \kappa_0, t \in Q$, where interval Q has length not less than κ_1 . The theorem is proved.

Remark 3.1 In [41] the author considers the following initial value problem for a system of impulsive differential equations:

$$\begin{aligned} x' &= A(t)x + G, t \neq \tau_i(m), \\ \Delta x|_{t=\tau_i(m)} &= I, \\ x(0) &= x_0, \end{aligned} \tag{15}$$

where $t \in [0, \infty), x \in \mathbb{R}^n$, the vector x_0 is fixed, G and I are constant vectors, and

(D1) the associated homogeneous system $x' = A(t)x$ is uniformly exponentially stable;

(D2) an impulsive interval sequence $\tau_i(m), i = 1, 2, \dots$, is defined by the formula

$$\tau_i = \sum_{k=1}^i (m_k + \theta),$$

where $m_k, k \geq 1$, are iterations of a sensitive map h defined on the bounded interval $\Xi \subset \mathbb{R}$, such that there exists a number $\theta' > 0$ satisfying $\theta + \inf\{t : t \in \Xi\} > \theta' > 0$.

The following definition was given in the paper.

DEFINITION 3.9 [41] *The solution of (15) is said to be chaotic with respect to the impulsive interval sequence provided that (i) this solution is uniformly bounded on the interval $[0, \infty)$; (ii) there exist two positive numbers $\epsilon_0, \sigma > 0$, such that for any $m \in \Xi$ one can find a point $m^* \in \Xi$ arbitrarily close to m , and a positive number T satisfying*

$$\mu(\{t_0 \leq t \leq T : \|x(t, m) - x(t, m^*)\| \geq \sigma\}) \geq \epsilon_0,$$

where $\mu(\cdot)$ represents the Lebesgue measure of a given set.

The author proves that under certain conditions, including (D1), (D2), the solution of (15) is chaotic in the sense of Definition 3.9.

Consider the following problem

$$\begin{aligned} x'(t) &= 0, t \neq \tau_i(m), \\ \Delta x|_{t=\tau_i(m)} &= (-1)^i, \\ x(0) &= x_0, \end{aligned} \tag{16}$$

where $t, x \in \mathbb{R}, x_0$ is fixed, and

$$\tau_i = \sum_{j=1}^i (m_j + 1), m_1 = m, m \in (0, 1), m_i = h(m_{i-1}), h(m) \equiv 0.$$

It is easily seen that $\tau_i = i + m - 1, i = 1, 2, \dots$. Consider two solutions of the problem, $x(t, m), x(t, m^*), m, m^* \in (0, 1), m \neq m^*$. Obviously, $\mu(\{0 < t < k : |x(t, m) - x(t, m^*)| \geq 1\}) \geq k|m - m^*|$, where k is a positive integer, and solution $x(t, m), m \in (0, 1)$, is bounded on $[0, \infty)$, as is $x(t, m^*)$. In other words, the solution is chaotic by Definition 3.9 even though there is no "chaotic" assumption on the map h , analogous to (D2), and no condition similar to (D1) is imposed on the system. So, this example shows that a more scrupulous investigation of the problem is required. In particular, not only sensitivity, but also other ingredients

of chaos should be involved in the discussion. We additionally emphasize that sensitivity in [41] is defined with respect to the impulsive interval sequence, or, it is better to say, with respect to the parameter $m \in \Xi$. Consequently, it cannot be considered a chaotic property. Our definition of sensitiveness involves the initial data, and it is natural for dynamics [7, 12, 20]. Finally, we should remark that system (16) is not sensitive in the sense of Definition 3.5.

3.3 The period-doubling cascade

Now we consider $\mu > 0$, μ being the parameter for the logistic map.

Since we use the logistic map as a tool to create the chaos, we expect to observe the period-doubling cascade.

Consider the logistic map $h(t, \mu) \equiv \mu t(1 - t)$, assuming this time that $\mu < 4$. It is known [10], that there exists an infinite sequence $3 < \mu_1 < \mu_2 < \dots < \mu_k \dots < 3.8284\dots$ such that $h(t, \mu_i), i \geq 1$, has an asymptotically stable prime period- 2^i point t_i^* with a region of attraction $(t_i^* - \delta_i, t_i^* + \delta_i)$. And beyond the value $3.8284\dots$, there are cycles with every integer period [9].

By the results of our paper the period-doubling process is occurring in the bounded region $\|x\| < M_1$ of the space \mathbb{R}^n , common for all $\mu > 0$. The chaotic properties are observed in the region, and, by Theorem 2.2, bounded solutions from this domain attract all other solutions. Moreover, the cascade generates infinitely many periodic solutions.

EXAMPLE 3.1 *Consider the following initial value problem*

$$\begin{aligned} x_1' &= 2/5x_2 + l \sin^2 x_2, \\ x_2' &= 2/5x_1 + l \sin^2 x_1, t \neq \zeta_i(t_0), \\ \Delta x_1|_{t=\zeta_i(t_0)} &= -\frac{4}{3}x_1, \\ \Delta x_2|_{t=\zeta_i(t_0)} &= -\frac{4}{3}x_2 + W(x_2), \end{aligned} \tag{17}$$

where $W(s) = 1 + s^2$, if $|s| \leq l$, l is a positive constant, and $W(s) = 1 + l^2$, if $|s| > l$. One can easily see that all the functions are lipschitzian with a constant proportional to l . The matrices of coefficients are

$$A = \begin{pmatrix} 0 & 2/5 \\ 2/5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -4/3 & 0 \\ 0 & -4/3 \end{pmatrix}.$$

The matrices commute, and the eigenvalues of the matrix

$$A + Ln(\mathcal{I} + B) = \begin{pmatrix} -\ln 3 & 2/5 \\ 2/5 & -\ln 3 \end{pmatrix}$$

are negative: $\lambda_{1,2} = -\ln 3 \pm 2/5 < 0$.

Let us check if condition (C3) holds. It is sufficient to verify that $-\frac{1}{2}s + W(s) \neq 0$, for all $s \in \mathbb{R}$.

If $|s| \leq l$, then $-\frac{1}{2}s + W(s) = s^2 - \frac{1}{2}s + 1$, and it is never equal to zero.

If $|s| > l$, then $-\frac{1}{2}s + W(s) = l^2 - \frac{1}{2}s + 1$. For the last expression to be zero, we need, $s = -2(1 + l^2)$, that is $|s| > l$. We have a contradiction. Thus, condition (C3) is valid. All the other conditions required by the Theorems could be easily checked with sufficiently small coefficient l . That is for $\mu > 4$ there is the chaotic behavior.

The numerical simulation of the chaos is not an easy task since even the verification of sensitivity requires two close values of the initial moment in the Cantor set Λ , which cannot be found easily. Hammel et al. [42] have given a computer-assisted proof that an approximate trajectory of the logistic map can be shadowed by a true trajectory for a long time. This result and the continuous dependence of the solutions on the sequence of discontinuity points make possible the following appropriate simulations.

In what follows we will demonstrate for $\mu = 4$ and $\mu = 3.8282$ the sensitivity and intermittency respectively.

Assume that $l = 10^{-2}$. Choose $\mu = 4$ in (17) and consider two solutions $x(t) = (x_1, x_2)$, $\bar{x}(t) = (\bar{x}_1, \bar{x}_2)$, with initial moments $t_0 = 7/9$ and $\bar{t}_0 = 7/9 + 3^{-12}$,

respectively. That is, we take the initial values close to each other, and, moreover, the solutions with identical initial values, $x(t_0) = \bar{x}(\bar{t}_0) = (0.005, 0.002)$. The graphs of the coordinates of these solutions (Fig. 1) show that the solutions abruptly become different when t is between 15 and 20, despite being very close to each other for all t in the interval $(\bar{t}_0, 15)$. One can conclude that the phenomenon of sensitivity is numerically observable.

Moreover, if one consider the sequence $(x_1(n), x_2(n)), n = 1, 2, 3, \dots, 75000$, in x_1, x_2 -plane (Fig. 2), then the chaotic attractor can be seen.

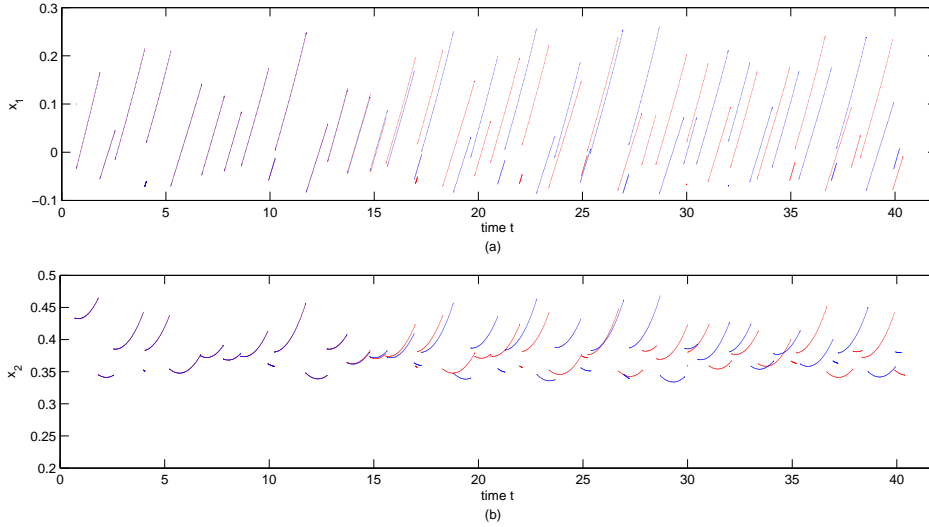


Figure 1: Simulation results. Graphs of coordinates $x_1(t), x_2(t)$ are in the blue, and of coordinates $\bar{x}_1(t), \bar{x}_2(t)$ are red. The coordinates in (a) and (b) abruptly become significantly different when t is near 15, while almost coinciding for all t in the interval $(t_1, 15)$.

Consider $\mu = 3.8282$ in (17). The coefficient's value of 3.8282 is such that the logistic map admits intermittency [43]. Fix $t_0 = 0.5$ and take a solution $(x_1(t), x_2(t))$ of the last system with the initial condition $x_1(t_0) = 0.002, x_2(t_0) = 0.005$. The result of simulation can be seen in Fig.2.

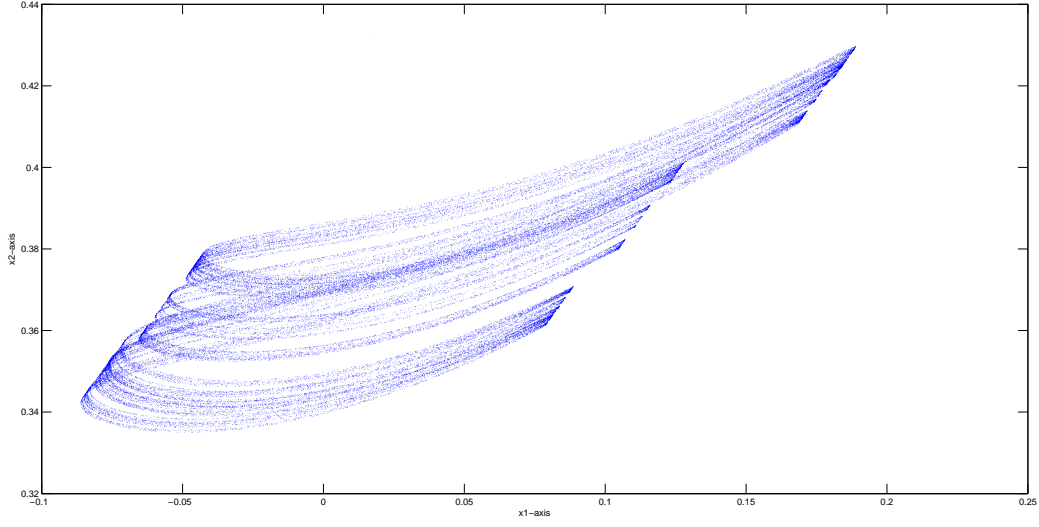


Figure 2: The chaotic attractor by a stroboscopic sequence $(x_1(n), x_2(n)), 1 \leq n \leq 75000$, is observable.

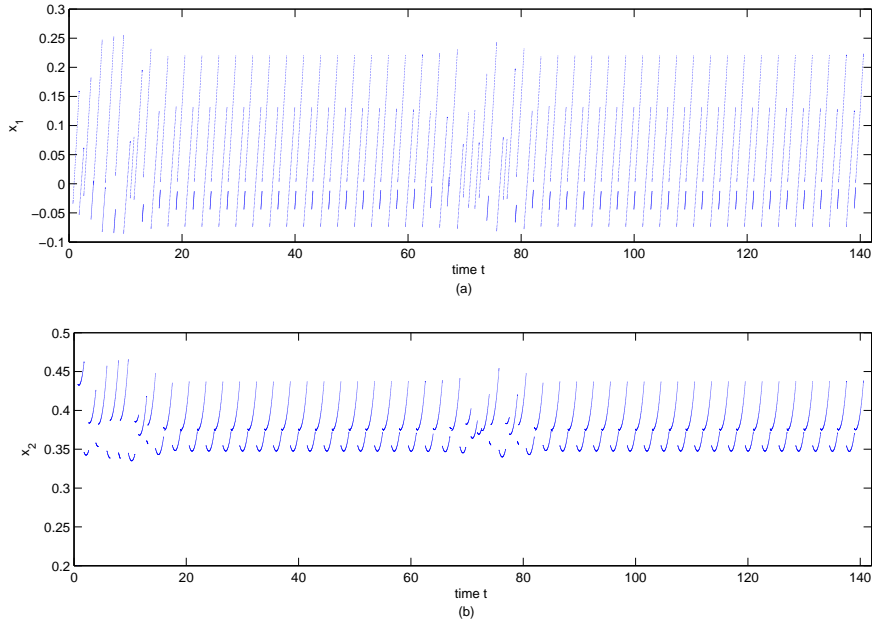


Figure 3: Simulation results. In (a) and (b) for coordinates x_1 and x_2 respectively the intermittency phenomenon is indicated.

4 Conclusion

The complex dynamics is obtained using Devaney's definition for guidance. We also prove the existence of a chaotic attractor consisting of solutions bounded on the whole real axis, and observe the period-doubling cascade for the problem.

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